LEBANESE AMERICAN UNIVERSITY
Division of Computer Science and Mathematics

Calculus III

Exam II

Fall 2008 (December 11, 2008)

Name: KEY ______________________ **ID:** ______________________

Circle the name of your instructor: **Dr. Habre** **Dr. Hamdan:**

Question Number	Grade
1. 8 %	
2. 8%	
3. 24%	
4. 24%	
5. 9%	
6. 27%	
Total	

1. (8%) Examine convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{3^n}$ and determine its sum.

$\sum_{n=1}^{\infty} \left(-\frac{2}{3}\right)^n$: geometric series with $r = -\frac{2}{3}$;

$|r| < 1 \Rightarrow$ series converges to: $\frac{1^{st} \text{ term}}{1 - r}$

$$= \frac{-\frac{2}{3}}{1 + \frac{2}{3}} = \boxed{\frac{-2}{5}}.$$

2. (8%) Consider the series: $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$.

(a) Find the third and fourth partial sums: s_3 and s_4.

$$S_3 = \frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} = \frac{1}{2}\left(1 - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$= \frac{1}{2}\left[1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \frac{1}{4} + \cancel{\frac{1}{3}} - \frac{1}{5}\right] = \frac{21}{40}$$

$$S_4 = \frac{21}{40} + \frac{1}{2}\left(\frac{1}{4} - \frac{1}{6}\right) = \frac{21}{40} + \frac{1}{2}\left(\frac{3-2}{12}\right) = \frac{21}{40} + \frac{1}{24} = \frac{63+5}{120} = \frac{68}{120}$$

$$= 17/30.$$

(b) Find the *nth* partial sum s_n, and then deduce the sum of the series.

$$S_n = \frac{1}{1.3} + \frac{1}{2.4} + \frac{1}{3.5} + \cdots + \frac{1}{n.(n+2)}$$

$$= \frac{1}{2}\left[1 - \cancel{\frac{1}{3}}\right] + \frac{1}{2}\left[\frac{1}{2} - \cancel{\frac{1}{4}}\right] + \frac{1}{2}\left[\cancel{\frac{1}{3}} - \frac{1}{5}\right] + \cdots + \frac{1}{2}\left[\cancel{\frac{1}{n-1}} - \frac{1}{n+1}\right] + \frac{1}{2}\left[\cancel{\frac{1}{n}} - \frac{1}{n+2}\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right].$$

$$\lim_{n\to\infty} S_n = \frac{1}{2}\left[\frac{3}{2}\right] = \frac{3}{4}.$$

3. (24%) Determine the convergence or divergence of the following series:

(a) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Ratio test: $\rho = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \frac{(n+1)\,n^n}{(n+1)^{n+1}}$

$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n}$

$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1 \quad \therefore$ series converges

(b) $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$: $\ln(n) < n^{0.5}$

$\Rightarrow \frac{\ln n}{n^2} < \frac{n^{0.5}}{n^2} = \frac{1}{n^{1.5}}$

But $\sum \frac{1}{n^{1.5}}$ converges (p-series, $p > 1$)

$\therefore$ By Direct Comparison test, $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges

(c) $\sum_{n=1}^{\infty} \frac{n}{e^n+5}$: $\frac{n}{e^n+5} < \frac{n}{e^n} = n e^{-n}$.

Now, we use integral test for $\sum_{n=1}^{\infty} n e^{-n}$:

$f(x) = x e^{-x}$; $x > 1$; $f(x) \geq 0$ and $f(x)$ is continuous

$\int_1^{\infty} x e^{-x} dx = \lim_{b\to\infty} \int_1^b x e^{-x} dx$; let $u = x$; $dv = e^{-x} dx$
$du = dx$; $v = -e^{-x}$

$\therefore \int x e^{-x} dx = -x e^{-x} + \int e^{-x} dx = -x e^{-x} - e^{-x}$

$\therefore \int_1^b x e^{-x} dx = \lim_{b\to\infty} [\underbrace{-b e^{-b}}_{0} - \underbrace{e^{-b}}_{0} + e^{-1} + e^{-1}]$ $\therefore$ converges. Thus by integral test, series converges.

4. (24%) Determine whether the following series converge absolutely, conditionally, or diverge:

(a) $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2+2n+2}$ $\rightarrow \sum \left|(-1)^n \frac{1}{n^2+2n+2}\right| = \sum \frac{1}{n^2+2n+2}$

LCT with $\sum \frac{1}{n^2}$ which converges ✓

Now, $\lim_{n\to\infty} \frac{\frac{1}{n^2+2n+2}}{\frac{1}{n^2}} = 1$ ∴ both series converge ✓ or both diverge ✓

∴ by LCT, $\sum(-1)^n \frac{1}{n^2+2n+2}$ converges absolutely. ✓

(b) $\sum_{n=1}^{\infty}(-1)^n \left(1-\frac{1}{n}\right)^{2n}$

$\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^{2n} = e^2 \neq 0$ ∴ series diverges

(c) $\sum_{n=2}^{\infty}(-1)^n \left(\frac{1}{n^{1/3}+7}\right)$:

Leibniz theorem:

$a_n = \frac{1}{n^{1/3}+7}$; $\lim_{n\to\infty} a_n = 0$ ✓

$\frac{1}{n^{1/3}+7} > \frac{1}{(n+1)^{1/3}+7}$ ∴

by Leibniz theorem, series converges.

But $\sum \frac{1}{n^{1/3}+7}$ diverges : LCT with $\sum \frac{1}{n^{1/3}}$

hence series converges absolutely.

5. (9%) Find the values of x for which the power series $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$ converges: Explain.

$$\sum_{n=1}^{\infty} \frac{|x|^n}{n\,3^n} : \quad \rho = \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n\,3^n}{|x|^n} = \lim_{n\to\infty} \frac{n|x|}{(n+1)\cdot 3} = \frac{|x|}{3}$$

$$\rho < 1 \Rightarrow -3 < x < 3 .$$

Now, $x = 3 \to \sum \frac{3^n}{n\,3^n} = \sum \frac{1}{n}$, diverges

$x = -3 \longrightarrow \sum \frac{(-1)^n}{n} \longrightarrow$ converges

$\therefore$ interval of convergence is: $-3 \leq 3 < 3.$

6. (27%) Using Maclaurin series, answer the following questions:

(a) Find $\lim_{x\to\infty} x^2(e^{10/x^2} - 1)$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{\frac{10}{x^2}} = \sum_{n=0}^{\infty} \frac{\left(\frac{10}{x^2}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{10^n}{x^{2n}\cdot n!}$$

$$\Rightarrow e^{10/x^2} = 1 + \sum_{n=1}^{\infty} \frac{10^n}{x^{2n}\cdot n!} \Rightarrow e^{10/x^2} - 1 = \sum_{n=1}^{\infty} \frac{10^n}{x^{2n}\cdot 2n!}$$

$$\therefore (e^{10/x^2} - 1)\cdot x^2 = x^2 \cdot \left[\frac{10}{x^2 \cdot 2!} + \frac{10^2}{x^4 \cdot 4!} + \frac{10^2}{x^6 \cdot 6!} + \cdots \right]$$

$$= \frac{10}{2!} + \frac{10^2}{x^2 \cdot 4!} + \frac{10^2}{x^4 \cdot 6!} + \cdots$$

$$\therefore \lim_{x\to\infty} x^2 (e^{10/x^2} - 1) = \frac{10}{2!} = 5.$$

(b) Represent the function $f(x) = \dfrac{x^2}{1-6x}$ as a power series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \; ; -1 < x < 1.$$

$$\therefore \frac{1}{1-6x} = \sum_{n=0}^{\infty} (6x)^n = \sum_{n=0}^{\infty} 6^n x^n \; ; \; -\frac{1}{6} < x < \frac{1}{6}.$$

$$\Rightarrow \frac{x^2}{1-6x} = \sum_{n=0}^{\infty} 6^n x^{n+2} \; ; \; -\frac{1}{6} < x < \frac{1}{6}$$

(c) Write the integral $\int e^{-x^2 dx}$ as a power series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \; ; \Rightarrow e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

$$\therefore \int e^{-x^2} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int x^{2n} dx.$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{x^{2n+1}}{2n+1} \left(+ C \right). \; ; \; x \in \mathbb{R}.$$